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A THEORETICAL TREATMENT OF CYCLIC PHENOMENA WITH PERIODIC PHASE--ETC(U)

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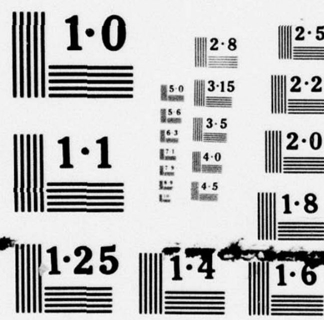

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(6) A THEORETICAL TREATMENT OF CYCLIC PHENOMENA WITH PERIODIC PHASE DISCONTINUITIES, PART I.
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JOB

Foreward

This memorandum covers only a small phase of the problem of instrumentation for signal enhancement research, and has been prepared primarily for internal distribution to aid others at NEL who may be interested in related problems. [redacted]. Work to January 1958 is covered.

Introduction

Recent signal processing techniques, exemplified by the DELTIC[1] and the ARMS[2] type of instrumentation, achieve a reduction in processing time by speeding up

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[1] Victor C. Anderson, "Deltic Correlator," Harvard Acoustics Lab., TM No. 37, Jan. 5, 1956.

[2] E. C. Westerfield, "Automatic Recycling Multiple Sampler," NEL TM No. 132, 17 Aug. 1955.

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and recycling an extensive section of the signal. In normal ARMS or DELTIC operation a primary unit is used to speed up the sampled signal. The sped-up section of signal may then be transferred to a storage unit for recycling. In the storage unit this sped-up section of the signal may be recycled repeatedly without any essential alteration until the processing is completed. The storage unit is then cleared and a new section of the sped-up signal is introduced from the primary unit and the process repeated. In the operation of the primary unit the essential difference from that of the storage unit is that each time the signal in the primary unit is recycled the oldest sample is dropped and a new sample added so that the section being circulated at any time consists of the most recently received signal samples. For the cross-correlation of two signals, the output of the storage unit on one signal may be multiplied by the output of a primary unit on the other signal.

It is the purpose of the present paper to determine the spectra for the outputs of the primary and of the storage units as well as for the correlator output. Although developed primarily in terms of the ARMS units the results are also applicable to the DELTIC Correlator and to other systems of this general nature. The method used for determining the correlator output is also applicable to the Sequential Matched Filter[3,4] in spite of its somewhat different mode of operation. For one mode of operation of the Sequential Matched Filter the results are the same

as for the normal ARMS Correlator. The present paper is Part I of a series of papers covering the theory of operation of this type of equipment. Topics to be covered in future memoranda in this series are:

Application to Single SideBand Signals,

Effect of Doppler,

Treatment of Large Doppler,

Effect of Finite Duration of Sample,

Lower SideBand and Double SideBand Operation,

Effect of Infinite Clipping,

Effect of Self Noise.

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[3]E. C. Westerfield, NEL LORAD Summary Report No. 698, Article 27, "Digital Techniques for Rapid Processing of Signals," p. VI-35, 22 June 1956.

[4]W. B. Allen and F. J. Smith, "Instrumentation of Sequentially Shifted Shifting Register," NEL TM No. 238, 9 Apr 1957.

Thanks are due to LT N. D. Harding, Jr., USN (at NEL on Temporary Additional Duty from U. S. Naval Post-Graduate School, Monterey, California) as well as to the regular members of section 2234 for careful reading and criticism of Part I.

Mathematical Background

When a waveform is expressed as a function of time, the Fourier Transform of this function expresses the complex frequency-spectrum of the waveform. Thus, using a lower case letter to denote a function and the corresponding capital letter to represent its Fourier Transform, the spectrum of the waveform u(t) may be written

$$\text{spec } u(t) = U(f) = \int_{-\infty}^{\infty} u(t) \exp(-2\pi i f t) dt, \qquad \text{1-1}$$

the functional notation "spec" being used here as an abbreviation of "spectrum of". In turn, the waveform function is itself given by the inverse transformation

$$u(t) = \int_{-\infty}^{\infty} U(f) \exp(2\pi i f t) df = \text{wav } U(f). \qquad \text{1-2}$$

It follows that the theory of Fourier Transforms is essentially a theory of waveform and spectrum analysis when referred to time and frequency coordinates, though of course having wider applications.

There doesn't seem to be any general agreement on a notation to represent the convolution of a pair of functions. American usage seems to favor an elevated asterisk between the function symbols, while Woodward[5], possibly following a British custom, uses a five-pointed star. In the present paper the asterisk is used to denote the complex conjugate of a function and the functional abbreviation "conv" signifies "convolution of". Thus the convolution of the functions u(t) and v(t) may be written

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[5] P. M. Woodward, "Probability and Information Theory", Chapter 2, (McGraw-Hill, 1953).

$$\begin{aligned} \text{conv}(u,v) &= \int_{-\infty}^{\infty} u(t_1)v(t-t_1)dt_1 \\ &= \int_{-\infty}^{\infty} u(t-t_1)v(t_1)dt_1 \\ &= \text{conv}(v,u) \end{aligned} \qquad \text{1-3}$$

representing, for example, the output of a linear filter with input u(t) and impulsive response v(t). One application of this operation which will occur frequently

in the present paper is the convolution of u(t) and the unit impulse function $\delta(t)$. ($\delta(t)$ is also known as the Dirac delta function.) Thus

$$\text{conv}\ \left[u(t), \delta(t)\right] = \int_{-\infty}^{\infty} u(t_1)\ \delta(t-t_1)dt_1 = u(t), \qquad 1\text{-}4$$

or, more generally,

$$\text{conv}\ \left[u(t),\ \delta(t-\tau)\right] = \int_{-\infty}^{\infty} u(t_1)\ \delta(t-t_1-\tau)dt_1 = u(t-\tau). \qquad 1\text{-}5$$

Actually it is the corresponding convolution in the frequency domain

$$\text{conv}\ \left[U(f),\ \delta(f-\phi)\right] = U(f-\phi) \qquad 1\text{-}6$$

which will find greatest application in the present paper. In practical applications it is not necessary to assume $\delta(t)$ to be infinite at t = 0. It may be assumed to have a width ϵ and height $1/\epsilon$, where ϵ is small compared to the shortest wavelengths involved, similarly for $\delta(f)$.

In his treatment of waveform analysis Woodward[5] has introduced some novel functional notations which help systematise the application of the method of Fourier Transforms to the determination of signal spectra. He defines four functions as follows:

$$\text{rect}\ t = 1,\ |t|<1/2$$
$$\quad\quad 0,\ |t|>1/2 \qquad 1\text{-}7$$

$$\text{sinc}\ f = (1/\pi f)\ \sin \pi f, \qquad 1\text{-}8$$

$$\text{rep}_R\ u(t) = \Sigma_n\ u(t-nR), \qquad 1\text{-}9$$

(where Σ_n signifies $\sum_{n=-\infty}^{\infty}$), and

$$\text{comb}_R\ u(t) = \Sigma_n\ \left[u(nR)\ \delta(t-nR)\right]. \qquad 1\text{-}10$$

For finite values of u(t) and u(nR) the summation in equation (1-10) may be written

$$\Sigma_n\ \left[u(nR)\ \delta(t-nR)\right] = u(t)\ \ \Sigma_n\ \delta(t-nR) \qquad 1\text{-}11$$

giving

$$\text{comb}_R\ u(t) = u(t)\ \ \text{rep}_R\ \delta(t). \qquad 1\text{-}12$$

As an example of the use of functions (1-7) and (1-8), the spectrum of a rectangular pulse of unit amplitude and unit duration centered at the origin may be written

$$\text{spec rect } t = \int_{-\infty}^{\infty} \text{rect } t \exp(-2\pi i f t)dt$$

$$= \int_{-\frac{1}{2}}^{\frac{1}{2}} \exp(-2\pi i f t)dt$$

$$= 2\int_{0}^{\frac{1}{2}} \cos 2\pi f t \; dt$$

$$= (1/\pi f)\int_{0}^{\pi f} \cos \theta \; d\theta$$

$$= (1/\pi f) \sin \pi f$$

$$= \text{sinc } f. \qquad \text{1-13}$$

Other simple basic spectra are

$$\text{spec } \delta(t) = \int_{-\infty}^{\infty} \delta(t) \exp(-2\pi i f t)dt$$

$$= \exp(0)$$

$$= 1 \qquad \text{1-14}$$

and

$$\text{spec } \delta(t-\tau) = \int_{-\infty}^{\infty} \delta(t-\tau) \exp(-2\pi i f t)dt$$

$$= \exp(-2\pi i f \tau) . \qquad \text{1-15}$$

Following Woodwards tabulation[4], a few of the basic operations on Fourier Transforms may be written:

$$\text{spec } U(t) = u(-f), \qquad \text{1-16}$$

$$\text{spec } u(t) = U(-f), \qquad \text{1-17}$$

$$\text{spec } u^*(t) = U^*(-f), \qquad \text{1-18}$$

$$\text{spec } u'(t) = 2\pi i f \; U(f), \quad (u' \equiv \frac{du}{dt}), \qquad \text{1-19}$$

$$\text{spec } u(t-\tau) = U(f) \exp(-2\pi i f \tau), \qquad \text{1-20}$$

$$\text{spec } u(t/T) = |T| \; U(fT), \qquad \text{1-21}$$

$$\text{spec } (Au + Bv) = AU + BV, \qquad \text{1-22}$$

$$\text{spec } uv = \text{conv } (U,V), \qquad \text{1-23}$$

$$\text{spec conv } (u,v) = U \; V, \qquad \text{1-24}$$

$$\text{spec rep}_R \; u(t) = |1/R| \; \text{comb}_{1/R} \; U(f), \qquad \text{1-25}$$

$$\text{spec comb}_R \; u(t) = |1/R| \; \text{rep}_{1/R} \; U(f). \qquad \text{1-26}$$

From (1-12), equation (1-25) may be written

$$\text{spec rep}_R\, u(t) = |1/R|\, U(f)\, \text{rep}_{1/R}\, \delta(f) \qquad 1\text{-}27$$

and (1-26) becomes

$$\text{spec}\left[u(t)\, \text{rep}_R\, \delta(t)\right] = |1/R|\, \text{rep}_{1/R}\, U(f). \qquad 1\text{-}28$$

Combinations of the above basic operations give other useful operations. For example, combination of (1-16) with (1-17) gives

$$\text{spec } U(-t) = u(f) \qquad 1\text{-}29$$

and combination of (1-20) and (1-21) gives

$$\text{spec } u\left[(t-\tau)/T\right] = |T|\, U(fT) \exp(-2\pi i f\tau). \qquad 1\text{-}30$$

Similarly, application of the operations to simple basic spectra such as those of (1-13), (1-14) and (1-15) may lead to other spectra of interest. As an example, application of (1-29) to (1-15) leads to

$$\text{spec } \exp 2\pi i \phi t = \delta(f-\phi) \qquad 1\text{-}31$$

Taking $\phi = 0$, this in turn gives

$$\text{spec } 1(t) = \delta(f). \qquad 1\text{-}32$$

Application of (1-20) or (1-22) to equation (1-31) gives

$$\text{spec } \exp(2\pi i\phi t + i\beta) = \exp(i\beta)\, \delta(f-\phi). \qquad 1\text{-}33$$

Combined with the identity

$$\cos\theta = (1/2)\left[\exp(i\theta) + \exp(-i\theta)\right] \qquad 1\text{-}34$$

this leads to

$$\text{spec } \cos(2\pi\phi t+\beta) = (1/2)\left[\delta(f-\phi)\exp(i\beta) + \delta(f+\phi)\exp(-i\beta)\right]. \qquad 1\text{-}35$$

Similarly, from the identity

$$\cos^2\theta = (1/2)(1+\cos 2\theta) \qquad 1\text{-}36$$

the spectrum of $\cos^2 \pi\phi t$ may be written

$$\text{spec } \cos^2\pi\phi t = (1/2)\, \text{spec}(1+\cos 2\pi\phi t) = (1/4)\left[\delta(f-\phi) + 2\delta(f) + \delta(f+\phi)\right]. \qquad 1\text{-}37$$

As another useful example, application of (1-21) to (1-13) gives

$$\text{spec rect}(t/T) = T\ \text{sinc}(fT) \qquad 1\text{-}38$$

Equation (1-31) may also be used to obtain a useful special form of (1-23)

$$\text{spec}\ \left[u(t)\ \exp 2\pi i \phi t\right] = \text{conv}\ \left[U(f),\ \text{spec}\ \exp 2\pi i \phi t\right]$$

$$= \text{conv}\ \left[U(f),\ \delta(f-\phi)\right] = U(f-\phi), \tag{1-39}$$

The last step coming from equation (1-6). Applied to equation (1-13) for example, operation (1-39) gives

$$\text{spec}\ \left[\text{rect}\ t\ \exp 2\pi i \phi t\right] = \text{sinc}\ (f-\phi). \tag{1-40}$$

Or, more generally, from (1-38) and (1-33),

$$\text{spec}\ \left[\text{rect}\ (t/T)\ \exp(2\pi i \phi t + i\beta)\right] = |T|\ \text{sinc}\ (f-\phi)T\ \exp\ (i\beta). \tag{1-41}$$

Another interesting and useful relation may be obtained by applying operation (1-25) to (1-26). This gives

$$\text{spec rep}_R\ \text{comb}_S\ u(t) = |1/RS|\ \text{comb}_{1/R}\ \text{rep}_{1/S}\ U(f). \tag{1-42}$$

The converse relation is obtained by applying operation (1-26) to (1-25). Thus

$$\text{spec comb}_S\ \text{rep}_R\ u(t) = |1/RS|\ \text{rep}_{1/S}\ \text{comb}_{1/R}\ U(f). \tag{1-43}$$

The equivalent relations, obtained by replacing each comb function in equations (1-42) and (1-43) by its rep equivalent from (1-12), may be written

$$\text{spec rep}_R\ \left[u(t)\ \text{rep}_S\ \delta(t)\right] = |1/RS|\ \text{rep}_{1/S}\ U(f)\ \text{rep}_{1/R}\ \delta(f) \tag{1-44}$$

and

$$\text{spec}\ \left[\text{rep}_R\ u(t)\ \text{rep}_S\ \delta(t)\right] = |1/RS|\ \text{rep}_{1/S}\ \left[U(f)\ \text{rep}_{1/R}\ \delta(f)\right]. \tag{1-45}$$

Woodward[5] has indicated some of the above results as well as some others of equal interest.

Theoretical Development

The Fourier Transforms in the preceding section were not just casual examples. Most of them may be taken as preliminary steps in the synthesis of the relations to be developed in this and succeeding sections. As an additional step in this synthesis, the application of (1-27) to (1-41) gives

$$\text{spec rep}_R \left[\text{rect } t/T \exp 2\pi i \phi t\right] = |T/R| \text{ sinc } (f-\phi)T \text{ rep}_{1/R}\, \delta(f). \qquad 2\text{-}1$$

For simplicity the phase β is taken as zero in this section. Since $\exp i\beta$ is a constant multiplier for both waveform and spectrum, the correction for β different from zero is usually obvious. As a further step in the development, operation (1-39) may be applied to (2-1) to obtain

$$\text{spec } \left\{\exp 2\pi i \phi_2 t \text{ rep}_R \left[\text{rect } t/T \exp 2\pi i \phi_1 t\right]\right\}$$
$$= |T/R| \text{ sinc } (f-\phi_1-\phi_2)T \text{ rep}_{1/R}\, \delta(f-\phi_2). \qquad 2\text{-}2$$

Also, application of (1-44) to (1-41) gives

$$\text{spec rep}_R \left[\text{rect } t/T \exp 2\pi i \phi t \text{ rep}_S\, \delta(t)\right] = |T/RS| \text{ rep}_{1/S} \text{ sinc}(f-\phi)T$$
$$\text{rep}_{1/R}\, \delta(f). \qquad 2\text{-}3$$

And from (1-39) this leads to

$$\text{spec } \left\{\exp 2\pi i \phi_2 t \text{ rep}_R \left[\text{rect } t/T \exp 2\pi i \phi_1 t \text{ rep}_S\, \delta(t)\right]\right\}$$
$$= |T/RS| \text{ rep}_{1/S} \text{ sinc}(f-\phi_1-\phi_2)T \text{ rep}_{1/R}\, \delta(f-\phi_2). \qquad 2\text{-}4$$

Application of (1-23) and (1-13) to (2-1) gives

$$\text{spec } \left\{\text{rect } t/T \text{ rep}_S \left[\text{rect } t/Q \exp 2\pi i \phi t\right]\right\}$$
$$= |TQ/S| \text{ conv } \left[\text{sinc } fT, \text{ sinc } (f-\phi)Q \text{ rep}_{1/S}\, \delta(f)\right]$$
$$= |TQ/S| \int \text{sinc } (f-f_1)T \text{ sinc } (f_1-\phi)Q \text{ rep}_{1/S}\, \delta(f_1)\,df_1$$
$$= |TQ/S| \sum_n \left[\text{sinc } (f-n/S)T \text{ sinc } (\phi-n/S)Q\right]. \qquad 2\text{-}5$$

And from (1-39)

$$\text{spec } \left\{\text{rect } t/T \exp 2\pi i \phi_2 t \text{ rep}_S \left[\text{rect } t/Q \exp 2\pi i \phi_1 t\right]\right\}$$
$$= |TQ/S| \sum_n \left[\text{sinc } (f-\phi_2-n/S)T \text{ sinc } (\phi_1-n/S)Q\right]. \qquad 2\text{-}6$$

Application of (1-27) again gives

$$\text{spec rep}_R \left\{\text{rect } t/T \exp 2\pi i \phi_2 t \ \ \text{rep}_S \left[\text{rect } t/Q \exp 2\pi i \phi_1 t\right]\right\}$$

$$= |TQ/RS| \sum_n \text{ sinc } (f-\phi_2-n/S)T \text{ sinc } (\phi_1-n/S)Q] \ \ \text{rep}_{1/R}\, \delta(f). \qquad 2\text{-}7$$

And from (1-39)

$$\text{spec} \left\{\exp \ 2\pi i \phi_3 t \ \ \text{rep}_R \left[\text{rect } t/T \exp 2\pi i \phi_2 t \ \ \text{rep}_S \ (\text{rect } t/Q \exp 2\pi i \phi_1 t)\right]\right\}$$

$$= |TQ/RS| \sum_n \left[\text{sinc } (f-\phi_2-\phi_3-n/S)T \text{ sinc } (\phi_1-n/S)Q\right] \ \ \text{rep}_{1/R}\, \delta(f-\phi_3). \qquad 2\text{-}8$$

The definition of the rep function (1-9) may be used to write a number of useful identities. Thus

$$\text{rep}_R \left\{\text{rect } t/T \exp 2\pi i \phi_1 t\right\} = \sum_n \left\{\text{rect } (t-nR)/T \ \ \exp \left[2\pi i \phi_1 (t-nR)\right]\right\} \qquad 2\text{-}9$$

may be multiplied by exp $2\pi i \phi_2 t$ to obtain

$$\exp 2\pi i \phi_2 t \ \ \text{rep}_R \left\{\text{rect } t/T \exp 2\pi i \phi_1 t\right\} = \sum_n \left\{\text{rect } \frac{t-nR}{T} \exp 2\pi i \left[(\phi_1+\phi_2)t-\phi_1 nR\right]\right\}. \qquad 2\text{-}10$$

Similarly

$$\text{rep}_R \left\{\text{rect } t/T \exp \ 2\pi i \phi_1 t \ \ \text{rep}_S\, \delta(t)\right\} = \sum_n \left\{\text{rect } \frac{t-nR}{T} \ \exp 2\pi i \phi_1 (t-nR) \ \ \text{rep}_S\, \delta(t-nR)\right\} \qquad 2\text{-}11$$

may be multiplied through by exp $2\pi i \phi_2 t$ to obtain

$$\exp 2\pi i \phi_2 t \ \text{rep}_R \left\{\text{rect } t/T \exp 2\pi i \phi_1 t \ \ \text{rep}_S\, \delta(t)\right\}$$

$$= \sum_n \left\{\text{rect } \frac{t-nR}{T} \ \exp 2\pi i \left[(\phi_1+\phi_2)t-\phi_1 nR\right] \ \text{rep}_S\, \delta(t-nR)\right\}. \qquad 2\text{-}12$$

And

$$\text{rep}_R \left\{\text{rect } t/T \exp 2\pi i \phi_2 t \ \ \text{rep}_S \left[\text{rect } t/Q \exp 2\pi i \phi_1 t\right]\right\}$$

$$= \sum_n \left\{\text{rect } \frac{t-nR}{T} \ \exp 2\pi i \phi_2 (t-nR) \sum_r \left[\text{rect } \frac{t-rS-nR}{Q} \ \exp 2\pi i \phi_1 (t-rS-nR)\right]\right\} \qquad 2\text{-}13$$

when multiplied by exp $2\pi i \phi_3 t$ gives

$$\exp 2\pi i \phi_3 t \ \ \text{rep}_R \left\{\text{rect } t/T \exp 2\pi i \phi_2 t \ \ \text{rep}_S \left[\text{rect } t/Q \exp 2\pi i \phi_1 t\right]\right\}$$

$$= \sum_n \left\{\text{rect } \frac{t-nR}{T} \ \sum_r \left[\text{rect } \frac{t-nR-rS}{Q} \ \exp 2\pi i \left[(\phi_1+\phi_2+\phi_3)t-(\phi_1+\phi_2)nR-\phi_1 rS\right]\right]\right\} \qquad 2\text{-}14$$

From (2-10) it is seen that (2-2) may be written

$$\text{spec } \Sigma_n \left\{ \text{rect } \frac{t-nR}{T} \exp 2\pi i \left[(\varphi_1+\varphi_2)t - \varphi_1 nR \right] \right\}$$

$$= |T/R| \text{ sinc } (f-\varphi_1-\varphi_2)T \text{ rep}_{1/R}\, \delta(f-\varphi_2) \qquad 2\text{-}15$$

and from (2-12) (2-4) may be written in the form

$$\text{spec } \Sigma_n \left\{ \text{rect } \frac{t-nR}{T} \exp 2\pi i \left[(\varphi_1+\varphi_2)t - \varphi_1 nR \right] \text{ rep}_S\, \delta(t-nR) \right\}$$

$$= |T/RS| \text{ rep}_{1/S} \text{ sinc}(f-\varphi_1-\varphi_2)T \text{ rep}_{1/R}\, \delta(f-\varphi_2). \qquad 2\text{-}16$$

Taking $\varphi_a = \varphi_1 + \varphi_2$ and $\varphi_1 R = \varphi_a P$, it is seen that (2-15) takes the form

$$\text{spec } \Sigma_n \left\{ \text{rect } \frac{t-nR}{T} \exp 2\pi i\, \varphi_a (t-nP) \right\} = |T/R| \text{ sinc } (f-\varphi_a)T \text{ rep}_{1/R}\, \delta(f-\varphi_2) \qquad 2\text{-}17$$

and (2-16) takes the form

$$\text{spec } \Sigma_n \left\{ \text{rect } \frac{t-nR}{T} \exp 2\pi i\, \varphi_a (t-nP) \text{ rep}_S\, \delta(t-nR) \right\}$$

$$= |T/RS| \text{ rep}_{1/S} \text{ sinc } (f-\varphi_a)T \text{ rep}_{1/R}\, \delta(f-\varphi_2) \qquad 2\text{-}18$$

where in each case

$$\varphi_2 = \varphi_a(1-P/R). \qquad 2\text{-}19$$

Or, more generally, taking $\varphi_1+\varphi_2 = \varphi_a \pm \varphi_b$ and $\varphi_1 R = \varphi_a P \pm \varphi_b R$, equation (2-15) may be written in the form

$$\text{spec } \Sigma_n \left\{ \text{rect } \frac{t-nR}{T} \exp 2\pi i \left[\varphi_a(t-nP) \pm \varphi_b (t-nR) \right] \right\}$$

$$= |T/R| \text{ sinc } (f-\varphi_a \mp \varphi_b)T \text{ rep}_{1/R}\, \delta(f-\varphi_2) \qquad 2\text{-}20$$

and (2-16) takes the form

$$\text{spec } \Sigma_n \left\{ \text{rect } \frac{t-nR}{T} \exp 2\pi i \left[\varphi_a(t-nP) \pm \varphi_b(t-nR) \right] \text{ rep}_S\, \delta(t-nR) \right\}$$

$$= |T/RS| \text{ rep}_{1/S} \text{ sinc } (f-\varphi_a \mp \varphi_b)T \text{ rep}_{1/R}\, \delta(f-\varphi_2) \qquad 2\text{-}21$$

where as before

$$\varphi_2 = \varphi_a(1-P/R). \qquad 2\text{-}22$$

From (2-14) it is seen that (2-8) may be written in the form

$$\text{spec } \Sigma_n \left\{ \text{rect } \frac{t-nR}{T} \Sigma_r \left[\text{rect } \frac{t-nR-rS}{Q} \exp 2\pi i \left[(\varphi_1+\varphi_2+\varphi_3)t - (\varphi_1+\varphi_2)nR - \varphi_1 rS \right] \right] \right\}$$

$$= |TQ/RS| \Sigma_n \left[\text{sinc}(f-\varphi_2-\varphi_3-n/S)T \text{ sinc } (\varphi_1-n/S)Q \right] \text{rep}_{1/R}\, \delta(f-\varphi_3) \qquad 2\text{-}23$$

where the n in the first member is unrelated to the n in the second member of

the equation, and should perhaps be distinguished from it by the use of subscripts or a different letter. The same remark applies to the equations which follow.

For the particular case $\phi_3 = 0$ and $\phi_2 = -\phi_1 = \phi_b$ equation (2-23) reduces to

$$\text{spec}\, \Sigma_n \left\{ \text{rect}\,(t-nR)/T \;\; \Sigma_r \left[\text{rect}(t-nR-rS)/Q \,\exp 2\pi i \phi_b rS \right] \right\}$$

$$= \left| TQ/RS \right| \Sigma_n \left[\text{sinc}\,(f-\phi_b-n/S)T \;\text{sinc}\,(\phi_b + n/S)Q \right] \text{rep}_{1/R}\, \delta(f). \qquad 2\text{-}24$$

Similarly, for the case $\phi_2 + \phi_3 = -\phi_1 = \phi_a$ and $(\phi_1 + \phi_2)R = -\phi_a S$ equation (2-23) reduces to

$$\text{spec}\, \Sigma_n \left\{ \text{rect}(t-nR)/T \;\; \Sigma_r \left[\text{rect}(t-nR-rS)/Q \;\; \exp 2\pi i \phi_a (n+r)S \right] \right\}$$

$$= \left| TQ/RS \right| \Sigma_n \left[\text{sinc}\,(f-\phi_a-n/S)T \;\text{sinc}\,(\phi_a + n/S)Q \right] \text{rep}_{1/R}\, \delta(f-\phi_3) \qquad 2\text{-}25$$

where

$$\phi_3 = \phi_a \; S/R. \qquad 2\text{-}26$$

Also, for the more general case $\phi_2 + \phi_3 = -\phi_1 = \phi_a \pm \phi_b$ and $(\phi_1 + \phi_2)R = -\phi_a S$ equation (2-23) reduces to

$$\text{spec}\, \Sigma_n \left[\text{rect}\, \frac{t-nR}{T} \;\; \Sigma_r \left\{ \text{rect}\, \frac{t-nR-rS}{Q} \;\; \exp 2\pi i \left[\phi_a nS + (\phi_a \pm \phi_b) rS \right] \right\} \right]$$

$$= \left| TQ/RS \right| \;\Sigma_n \;\; \left[\text{sinc}\,(f-\phi_a \mp \phi_b - n/S)T \;\text{sinc}\,(\phi_a \pm \phi_b + n/S) \right] \;\text{rep}_{1/R}\, \delta(f-\phi_3) \qquad 2\text{-}27$$

where as before equation (2-26) relates ϕ_3 to ϕ_a.

Treatment of a Single-Frequency Input

Consider a reel of magnetic tape on which is recorded a continuous sinusoidal wave of frequency φ_s. If the tape is played back at N times the recording speed the playback frequency may be designated as

$$\varphi_b = N\varphi_s \tag{3-1}$$

Considering a particular section of the tape with playback time R, the sinusoidal wave can be represented by $\cos(2\pi\varphi_b t + \beta_b)$ when referred to the midpoint of the section as origin. If the ends of this section are now joined to form a continuous loop for playback, the resulting output waveform may be represented by the summation

$$\Sigma_n \left\{\mathrm{rect}\ \frac{t-nR}{R}\ \cos\left[2\pi\varphi_b(t-nR)+\beta_b\right]\right\} = \mathrm{rep}_R\left[\mathrm{rect}(t/R)\ \cos\ (2\pi\varphi_b t+\beta_b)\right] \tag{3-2}$$

and from identity (1-34), equation (2-1) and (1-33) may be used to obtain

$$\begin{aligned}\mathrm{spec\ rep}_R\left[\mathrm{rect}(t/R)\cos(2\pi\varphi_b t+\beta_b)\right] &= (1/2)\left[\mathrm{sinc}\ (f-\varphi_b)R\ \exp(i\beta_b)\right.\\ &\left.+\ \mathrm{sinc}\ (f+\varphi_b)R\ \exp(-i\beta_b)\right]\ \mathrm{rep}_{1/R}\ \delta(f)\end{aligned} \tag{3-3}$$

as the resulting output spectrum. When referred to an origin at the beginning of the section this equation gives results identical with those obtained by Spetner[4]

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[4]Lee M. Spetner, "Errors in Power Spectra due to Finite Sample", J. Appl. Phys., 25, No. 5, pp 653-659, (May 1954).

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for this type of recycling storage unit.

The operation of a digital type recycling unit such as the ARMS or DELTIC storage unit is essentially the same as that of the magnetic tape loop just discussed. One of the chief differences is that instead of a continuous waveform the ARMS and DELTIC units show only polarity, not magnitude. The treatment of this last effect will be reserved for a later paper. The effect of periodic sampling may be investigated with the aid of equation (2-19). Taking N as the number of stages in the ARMS storage unit, the recycling period for one mode of operation is the same as the

input sampling period

$$R = NS \qquad 3\text{-}4$$

where S is the period between successive samples on the output. This corresponds to the usual mode of operation of the ARMS storage unit, N being the number of stored samples. Corresponding to (3-2) the resulting output waveform may thus be represented by the summation

$$\Sigma_n \left\{ \text{rect}\, \frac{t-nR}{R} \cos\left[2\pi\phi_b(t-nR)+\beta_b\right] \text{rep}_S\, \delta(t-nR)\right\}$$
$$= \text{rep}_R \left[\text{rect}(t-R) \cos(2\pi\phi_b t+\beta_b)\, \text{rep}_S\, \delta(t)\right] \qquad 3\text{-}5$$

And application of identity (1-34) and (1-33) to (2-3) gives

$$\text{spec rep}_R \left[\text{rect}(t/R) \cos(2\pi\phi_b t+\beta_b)\text{rep}_S\, \delta(t)\right]$$
$$= \left|1/2S\right| \text{rep}_{1/S} \left[\text{sinc}(f-\phi_b)R \exp(i\beta_b)+\text{sinc}(f+\phi_b)R \exp(-i\beta_b)\right] \text{rep}_{1/R}\,\delta(f) \qquad 3\text{-}6$$

as the corresponding output spectrum. The case for samples of finite duration is reserved for treatment in a later section.

For the primary unit the mathematical formulation used in determining the output spectrum is somewhat more involved than that for the storage unit. The primary ARMS unit for one type of operation has only N-1 stages and the period between identical samples on the output is

$$P = (N-1)S = R-S. \qquad 3\text{-}7$$

The addition of a new sample in the place of the oldest sample, however, extends the consecutive section by one sample. Accordingly R is both the input sampling interval and the interval between phase discontinuities in the output waveform the same as for the corresponding storage unit. Selecting the midpoint between a pair of successive phase discontinuities as the origin, the output waveform for this recycling period may be represented by $\cos(2\pi\phi_a t+\beta_a)$. For the next recycling period it would then be $\left[2\pi\phi_a(t-P)+\beta_a\right]$, etc. Thus, neglecting the effect of sampling and limiting, the expression for the output waveform of the primary unit may be written

$$\Sigma_n \left\{\text{rect}\, \frac{t-nR}{R} \cos\left[2\pi\phi_a(t-nP)+\beta_a\right]\right\}. \qquad 3\text{-}8$$

And from (1-33), (1-34) and (2-17) the corresponding output spectrum would be

$$\text{spec}\ \Sigma_n \left\{ \text{rect}\ \frac{t-nR}{R}\ \cos\left[2\pi\phi_a(t-nP)+\beta_a\right]\right\}$$

$$= (1/2)\left[\text{sinc}(f-\phi_a)R\ \exp(i\,\beta_a)\text{rep}_{1/R}\ \delta(f-k\phi_a)\right.$$

$$\left.+\ \text{sinc}(f+\phi_a)R\ \exp(-i\,\beta_a)\ \text{rep}_{1/R}\ \delta(f+k\phi_a)\right] \qquad 3\text{-}9$$

where from (2-19), (3-4) and (3-7)

$$k = 1-P/R = 1 - \frac{R-S}{R} = \frac{S}{R} = 1/N \qquad 3\text{-}10$$

As in the case of the storage unit the effect of sampling is taken into account by introducing the sampling function $\text{rep}_S\,\delta(t)$.

From (1-33), (1-34) and (2-18) this gives

$$\text{spec}\ \Sigma_n \left\{ \text{rect}\ \frac{t-nR}{R}\ \cos\left[2\pi\phi_a(t-nP)+\beta_a\right]\ \text{rep}_S\ \delta(t-nR)\right\}$$

$$= \left|1/2S\right| \left\{\text{rep}_{1/S}\ \text{sinc}(f-\phi_a)R\ \exp(i\,\beta_a)\text{rep}_{1/R}\ \delta(f-\phi_a/N)\right.$$

$$\left.+\ \text{rep}_{1/S}\ \text{sinc}(f+\phi_a)R\ \exp(-i\,\beta_a)\ \text{rep}_{1/R}\ \delta(f+\phi_a/N)\right\} \qquad 3\text{-}11$$

as the corresponding output spectrum when sampling is taken into account. Since the speed-up factor of the ABMS unit is N, it is seen that the value ϕ_a/N in the δ function in the second member of equation (3-11) is the frequency in the input waveform in the primary channel prior to speed-up.

Comparison of equation (33) with (3-9), or (3-6) with (3-11), indicates that there are beat frequencies $r/R \pm \phi_a/N$, ($r = 0, \pm 1, \pm 2, \ldots\ldots$), between the output of the primary and that of the storage unit. Moreover, since one of the peak lines in the primary output spectrum is in the neighborhood of ϕ_a and one of those in the storage unit is in the neighborhood of ϕ_b, it is to be expected that some of the strongest lines in the product spectrum will be in the neighborhood of $\phi_b - \phi_a$ and $\phi_b + \phi_a$ but not necessarily precisely at these values as would be expected in the case of non-cyclic operations.

It is necessary at this point to distinguish between two different types of signal processing instrumentation employing speed-up. One type is represented by the ABMS or DELTIC Correlator, for which the outputs of the primary and storage units

are available independently for filtering prior to correlation. In the Sequential Matched Filter, illustrating the other type of speed-up instrumentation, the product function is obtained internally so that separate intermediate filtering of primary and storage outputs is not possible. For the first type of instrumentation it may be necessary to investigate the effects of this intermediate filtering process. The treatment of this type of operation is reserved for a later section. When no filtering or other modifying operation is performed on the individual outputs prior to formation of the product function, the operation of the ARMS or DELTIC Correlator is essentially equivalent to that of the Sequential Matched Filter. This type of operation may be investigated more directly by treating the output product function, starting with the product of equations (3-2) and (3-8),

$$\Sigma_n \left\{ \text{rect}\ \frac{t-nR}{R}\ \cos\left[2\pi\phi_a(t-nP)+\beta_a\right]\ \cos\left[2\pi\phi_b(t-nR)+\beta_b\right]\right\}$$

$$= (1/2)\ \Sigma_n\ \text{rect}\ \frac{t-nR}{R} \left\{\cos\left[2\pi\phi_a(t-nR)-2\pi\phi_b(t-nR)+\beta_a-\beta_b\right]\right.$$

$$\left. +\cos\left[2\pi\phi_a(t-nP)+2\pi\phi_b(t-nR)+\beta_a+\beta_b\right]\right\}. \qquad 3\text{-}12$$

From identity (1-34), equation (1-33) and (2-20) applied to (3-12) give

$$\text{spec}\ (1/2)\ \Sigma_n \left\{\text{rect}\ \frac{t-nR}{R}\ \cos\left[2\pi\phi_a(t-nP)-2\pi\phi_b(t-nR)+\beta_a-\beta_b\right]\right\}$$

$$= (1/4)\ \left[\text{sinc}(f-\phi_\delta)R\ \exp(i\beta_\delta)\ \text{rep}_{1/R}\ \delta(f-k\phi_a)\right.$$

$$\left. +\ \text{sinc}(f+\phi_\delta)R\ \exp(-i\beta_\delta)\ \text{rep}_{1/R}\ \delta(f+k\phi_a)\right] \qquad 3\text{-}13$$

for the corresponding difference-frequency spectrum, and

$$\text{spec}\ (1/2)\ \Sigma_n \left\{\text{rect}\ \frac{t-nR}{R}\ \cos\left[2\pi\phi_a(t-nP)+2\pi\phi_b(t-nR)+\beta_a+\beta_b\right]\right\}$$

$$= (1/4)\ \left[\text{sinc}(f-\phi_\sigma)R\ \exp(i\beta_\sigma)\ \text{rep}_{1/R}\ \delta(f-k\phi_a)\right.$$

$$\left. +\ \text{sinc}(f+\phi_\sigma)R\ \exp(-i\beta_\sigma)\ \text{rep}_{1/R}\ \delta(f+k\phi_a)\right] \qquad 3\text{-}14$$

for the corresponding sum-frequency spectrum,

where

$$\phi_\delta = \phi_a - \phi_b, \qquad 3\text{-}15$$

$$\beta_\delta = \beta_a - \beta_b, \qquad 3\text{-}16$$

$$\phi_\sigma = \phi_a + \phi_b, \qquad 3\text{-}17$$

and

$$\beta_\sigma = \beta_a + \beta_b. \qquad 3\text{-}18$$

From equation (2-21), the effect of sampling leads to

$$\left|1/4S\right| \Big[\mathrm{rep}_{1/S}\ \mathrm{sinc}(f-\phi_\delta)R\ \exp(i\beta_\delta)\ \mathrm{rep}_{1/R}\ \delta(f-k\phi_a)$$

$$+\ \mathrm{rep}_{1/S}\ \mathrm{sinc}(f+\phi_\delta)R\ \exp(-i\beta_\delta)\ \mathrm{rep}_{1/R}\ \delta(f+k\phi_a)\Big] \qquad 3\text{-}19$$

for the difference-frequency spectrum, and in similar fashion

$$\left|1/4S\right|\ \mathrm{rep}_{1/S}\ \mathrm{sinc}(f-\phi_\sigma)R\ \exp(i\beta_\sigma)\ \mathrm{rep}_{1/R}\ \delta(f-k\phi_a)$$

$$+\ \mathrm{rep}_{1/S}\ \mathrm{sinc}(f+\phi_\sigma)R\ \exp(-i\beta_\sigma)\ \mathrm{rep}_{1/R}\ \ (f+k\phi_a)\Big] \qquad 3\text{-}20$$

for the corresponding sum-frequency spectrum. Taken together equations (3-19) and (3-20) provide the complete spectrum of the output product function (3-12) when the effect of sampling is taken into account. In practice, however, the sum-frequency spectrum will usually be removed by filtering so that the output spectrum of interest is that provided by equation (3-19). It may be noted that except for the amplitude factor $1/S$ the zeroth order term of $\mathrm{rep}_{1/S}$ in (3-19) is identical with the specturm in (3-13). Since in the usual case $|\phi_\delta|$ will be less than 1/2S, it is seen that a low pass filter with cut-off at 1/2S on the output of (3-19) will effectively eliminate all except this zeroth order term thereby practically eliminating the effect of sampling. For present purposes, therefore, (3-13) may be taken as the spectrum of interest on the output of the multiplier. From equation (3-10) this spectrum may be written

$$(1/4)\ \Big[\mathrm{sinc}(f-\phi_\delta)R\ \ \exp(i\beta_\delta)\ \mathrm{rep}_{1/R}\ \delta(f-\phi_a/N)$$

$$+\ \mathrm{sinc}(f+\phi_\delta)R\ \exp(-i\beta_\delta)\ \mathrm{rep}_{1/R}\ \delta(f+\phi_a/N)\Big]. \qquad 3\text{-}21$$

It should be noted that ϕ_a/N appearing in the δ function in (3-21) is the frequency prior to speed-up in the channel containing the primary ARMS unit. If the cross product had been between the output of the primary unit on channel b and that of a storage unit on channel a, this term would have been replaced by ϕ_b/N.

The first term of equation (3-21) represents a line spectrum with frequency intervals 1/R between the lines as illustrated in figure 1 for a particular set of parameters. As was indicated from comparison of the outputs of the primary and storage ARMS units, the strongest lines in the difference frequency spectrum are in the neighborhood of

$\phi_\delta = \phi_a - \phi_b$. The integer corresponding to the strongest line is designated r_m in figure 1. From the definition of the rep function, equation (1-9), it is seen that spectrum (3-21) may be written in the form

$$(1/4)\Big[\mathrm{sinc}(f-\phi_\delta)R\,\exp(i\beta_\delta)\,\Sigma_r\,\delta(f-\phi_a/N-r/R)$$
$$+\,\mathrm{sinc}(f+\phi_\delta)R\,\exp(-i\beta_\delta)\,\Sigma_r\,\delta(f+\phi_a/N+r/R)\Big]$$
$$= (1/4)\,\Sigma_r\Big[\mathrm{sinc}(f-\phi_\delta)R\,\exp(i\beta_\delta)\,\delta(f-\phi_a/N-r/R)$$
$$+\,\mathrm{sinc}(f+\phi_\delta)R\,\exp(-i\beta_\delta)\,\delta(f+\phi_a/N+r/R)\Big]$$
$$= (1/4)\,\Sigma_r\Big[\mathrm{sinc}(\phi_a/N+r/R-\phi_\delta)R\,\exp(i\beta_\delta)\,\delta(f-\phi_a/N-r/R)$$
$$+\,\mathrm{sinc}(-\phi_a/N-r/R+\phi_\delta)R\,\exp(-i\beta_\delta)\,\delta(f+\phi_a/N+r/R)\Big]$$
$$= (1/4)\,\Sigma_r\Big\{\mathrm{sinc}(\phi_\delta-\phi_a/N-r/R)R\Big[\exp(i\beta_\delta)\,\delta(f-\phi_a/N-r/R$$
$$+\,\exp(-i\beta_\delta)\,\delta(f+\phi_a/N+r/R\Big]\Big\}$$
$$= (1/2)\,\Sigma_r\Big\{\mathrm{sinc}(\phi_a/N+r/R-\phi_\delta)R\,\mathrm{spec}\,\cos\big[2\pi(\phi_a/N+r/R)t+\beta_\delta\big]\Big\}$$
$$= (1/2)\,\mathrm{spec}\,\Sigma_r\Big\{\mathrm{sinc}(\phi_a/N+r/R-\phi_\delta)R\,\cos\big[2\pi(\phi_a/N+r/R)t+\beta_\delta,\big]\Big\}, \qquad 3\text{-}22$$

the last steps coming from (1-35) and (1-22).

It follows that the output waveform corresponding to the difference frequency spectrum can be represented by

$$(1/2)\,\Sigma_r\Big\{\mathrm{sinc}(\phi_a/N+r/R-\phi_\delta)R\,\cos\big[2\pi(\phi_a/N+r/R)t+\beta_\delta\big]\Big\}. \qquad 3\text{-}23$$

The smallest value of $|\phi_a/N + r/R - \phi_\delta|$ corresponds of course to the strongest line in the spectrum. Identifying the corresponding value of the integer r by the subscript m, it is seen that an ideal narrowband filter, of width not more than 1/R encampassing this line will give the single term

$$(1/2)\,\mathrm{sinc}(\phi_a/N+r_m/R-\phi_\delta)R\,\cos\big[2\pi(\phi_a/N+r_m/R)t+\beta_\delta\big] \qquad 3\text{-}24$$

ad the wavefrom of the filtered output. For ϕ_a less than 1/S, that is ϕ_a/N less than 1/R it is seen that this line will lie between r_m/R and $(r_m+1)/R$.

Similarly it is seen that the output waveform corresponding to the sum-frequency spectrum in (3-21) can be represented by

$$(1/2)\,\Sigma_r\Big\{\mathrm{sinc}(\phi_a/N+r/R-\phi_\sigma)R\,\cos\big[2\pi(\phi_a/N+r/R)t+\beta_\sigma\big]\Big\} \qquad 3\text{-}25$$

And an expression analagous to that in (3-24) can be written for the waveform of the principal line in this spectrum.

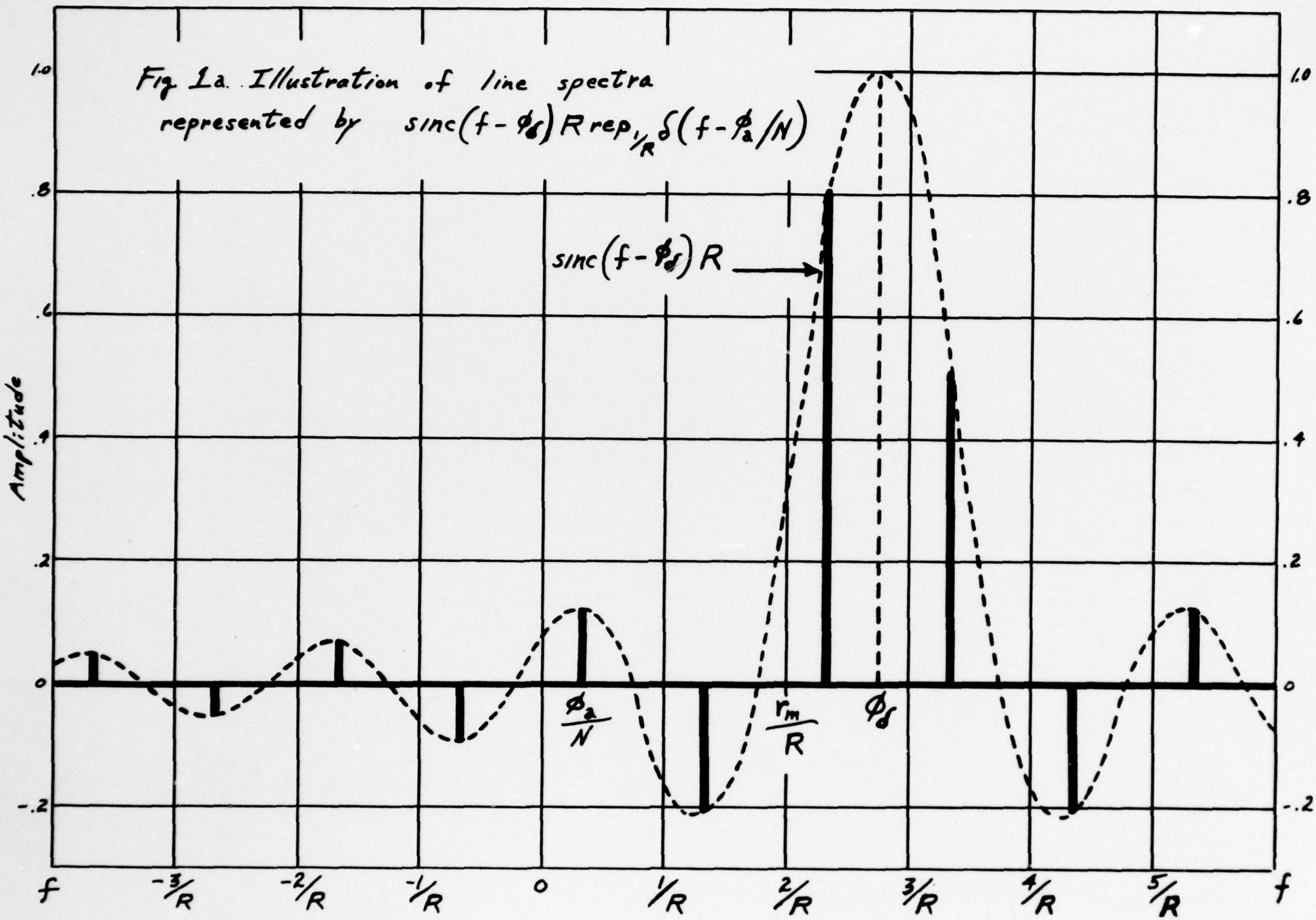

Fig 1a. Illustration of line spectra represented by $\mathrm{sinc}(f - \phi_\delta) R \, \mathrm{rep}_{1/R} \, \delta(f - \phi_a/N)$

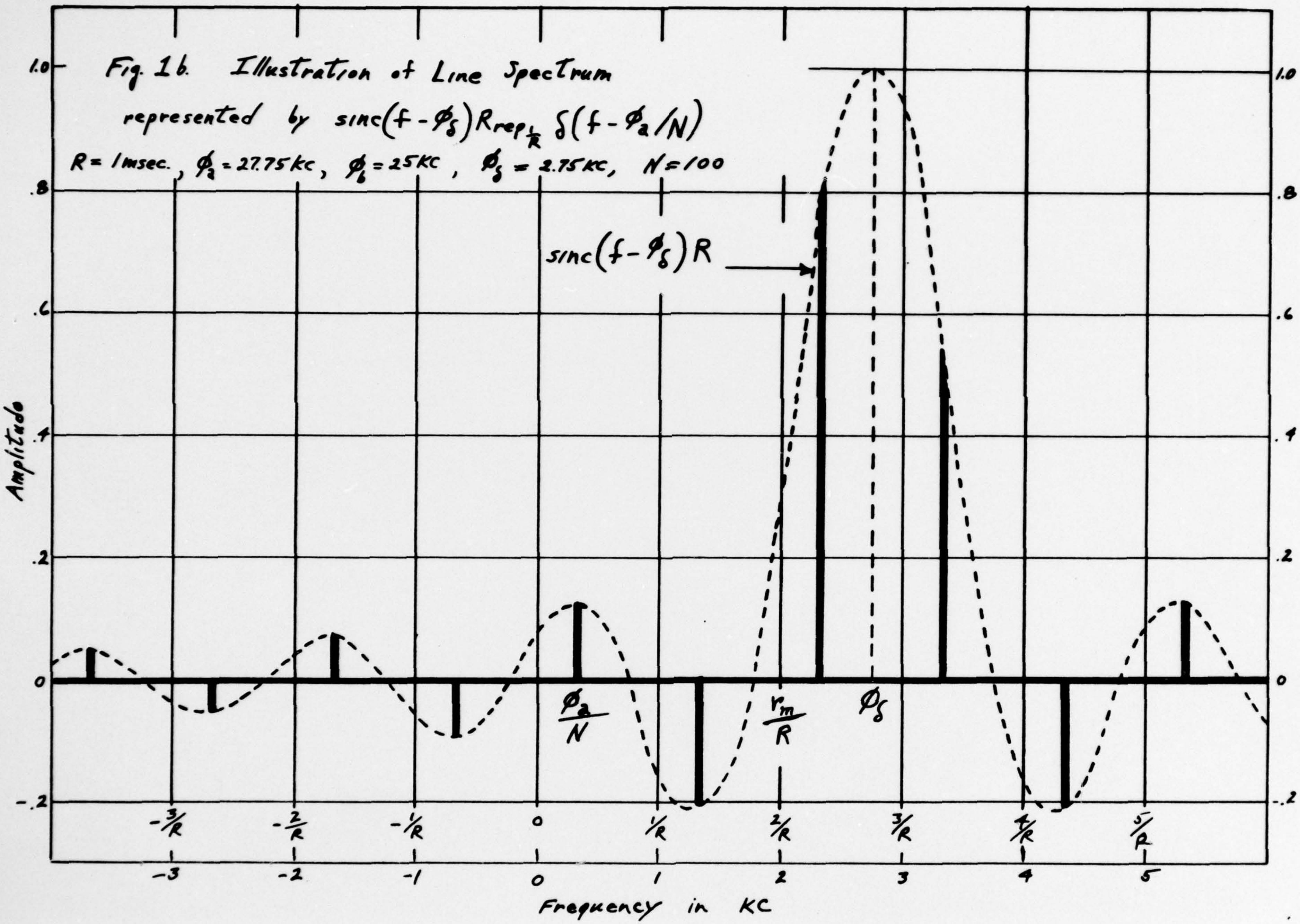

Fig. 1b. Illustration of Line Spectrum represented by $\mathrm{sinc}(f-\phi_\delta) R\,\mathrm{rep}_{\frac{1}{R}}\, \delta(f-\phi_a/N)$

$R = 1\,\mathrm{msec.}$, $\phi_a = 27.75\,\mathrm{kc}$, $\phi_b = 25\,\mathrm{KC}$, $\phi_\delta = 2.75\,\mathrm{KC}$, $N = 100$